

## SOBOLEV-TYPE EMBEDDINGS IN BESOV-MORREY SPACES WITH VARIABLE EXPONENTS

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**Abstract.** In the present paper, by using the semimodular theory, we prove the embeddings between Besov-Morrey spaces and Triebel-Lizorkin-Morrey spaces with variable exponents. We also give Sobolev-type embeddings in Besov-Morrey spaces with variable exponents.

**Keywords:** Besov-Morrey spaces, variable exponents, embeddings.

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## 1 Introduction

In the last decades a lot of researchers have given a great importance to the smoothness spaces built on the idea of Morrey spaces  $M_{p,u}$ . The Besov-Morrey spaces  $\mathcal{N}_{p,u,q}^\alpha$ , Triebel-Lizorkin-Morrey spaces  $\mathcal{E}_{p,q}^{\alpha,u}$  and Besov-type spaces  $B_{p,q}^{\alpha,\tau}$  are examples of that. The scales  $\mathcal{N}_{p,u,q}^\alpha$  were presented in Kozono et al. (1994) and developed later by Mazzucato et al. (2003). The spaces  $\mathcal{E}_{p,q}^{\alpha,u}$  were introduced by Tang et al. (2005) and  $B_{p,q}^{\alpha,\tau}$  were presented in El Baraka (2006). It is well known that when  $q$  is finite, the scales  $\mathcal{N}_{p,u,q}^\alpha$  and  $B_{p,q}^{\alpha,\tau}$  are different. However, both include the classical Besov spaces  $B_{p,q}^\alpha$  as a particular case. the spaces  $B_{p,q}^{\alpha,\tau}$  with  $q < \infty$ , also the classical Besov spaces is a particular case of  $B_{p,q}^{\alpha,\tau}$ .

Over the years, the function spaces with variable smoothness and integrability have attracted attention of many researchers not only by theoretical reasons but also by the role played by such spaces in some applications, including the modeling of field of electronic fluid mechanics, image restoration, optimization, etc. (See (Afraites et al., 2022; Laghrib et al., 2021; Nachaoui et al., 2021; Yacini et al., 2021))

Recently, Almeida & Caetano (2020) studied Morrey spaces with variable exponents  $M_{p(\cdot),u(\cdot)}$ , which generalize variable exponent Lebesgue space, see Remark 2 . Diening et al. (2009) introduced the Triebel-Lizorkin spaces with variable indices  $F_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$ . Many applications of these spaces were given, see Fu et al. (2011); Yuan et al. (2010). The Besov spaces  $B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$  have first been introduced in Almeida & Hästö (2010) by using the mixed Lebesgue-sequence spaces  $L_{p(\cdot)}(\ell_{q(\cdot)})$  and they proved the basic embeddings between Besov spaces and Triebel-Lizorkin spaces with variable smoothness and integrability.

$$B_{p(\cdot),q_0(\cdot)}^{\alpha(\cdot)} \hookrightarrow B_{p(\cdot),q_1(\cdot)}^{\alpha(\cdot)}, \quad \text{when } q_0 \leq q_1.$$

$$B_{p(\cdot),q_0(\cdot)}^{\alpha_0(\cdot)} \hookrightarrow B_{p(\cdot),q_1(\cdot)}^{\alpha_1(\cdot)}, \quad \text{when } (\alpha_0 - \alpha_1)^- > 0.$$

And

$$B_{p(\cdot),\min\{p(\cdot),q(\cdot)\}}^{\alpha(\cdot)} \hookrightarrow F_{p(\cdot),q(\cdot)}^{\alpha(\cdot)} \hookrightarrow B_{p(\cdot),\max\{p(\cdot),q(\cdot)\}}^{\alpha(\cdot)}, \quad \text{when } p^+, q^+ < \infty.$$

Also, the authors in Almeida & Hästö (2010) showed the Sobolev-type embeddings in the spaces  $B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$ , for more details we refer the reader to see Almeida & Hästö (2010). Recently, in Almeida & Caetano (2020) the authors presented an appropriate setting to the full scales  $\mathcal{N}_{p(\cdot),u(\cdot),q(\cdot)}^{\alpha(\cdot)}$  of Besov-Morrey spaces with variable exponents and also introduced mixed Morrey-sequence spaces. Later, António and Kempka introduced the Triebel-Lizorkin-Morrey spaces with variable exponents  $\mathcal{E}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),u(\cdot)}$  for more properties related to these spaces we refer the reader to see Besov (2003). Also, it is well known that if  $p(\cdot) = u(\cdot)$ , then Besov-Morrey spaces coincide with Besov spaces,  $\mathcal{N}_{p(\cdot),p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n) = B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$  (Besov-Morrey spaces larger than Besov spaces).

There is a rich literature about partial differential equations in different spaces with variable exponents, for instance, for Navier-Stokes equations and related models, we have well-posedness results in the critical case of the following spaces: Lebesgue space  $L^{p(\cdot)}$  Diening et al. (2011), Fourier-Besov spaces  $\mathcal{FB}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$  (Abidin et al., 2018) and Fourier-Besov-Morrey spaces  $\mathcal{FN}_{p(\cdot),u(\cdot),q(\cdot)}^{\alpha(\cdot)}$  (Abidin et al., 2021; Azanzal et al., 2021,a,b,c), among others.

This paper is organized as follows. In Section 2, we recall the definitions of modular spaces, variable exponents Lebesgue spaces, Morrey spaces, mixed Lebesgue-sequence spaces, mixed Morrey-sequence spaces, 2-microlocal Besov-Morrey spaces and 2-microlocal Triebel-Lizorkin-Morrey spaces and we present some properties about these spaces. Our results on basic embeddings between variable exponents Besov-Morrey and Triebel-Lizorkin-Morrey spaces also Sobolev-type embeddings are proved in Section 3.

## 2 Preliminaries

### 2.1 General notation

In this paragraph, we introduce some of general notations we use throughout the paper. We denote by  $\mathbb{R}^n$  the  $n$ -dimensional real Euclidean space,  $\mathbb{N}$  the set of all natural numbers and  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ .  $B(x, r)$  is the open ball in  $\mathbb{R}^n$  centered at  $x \in \mathbb{R}^n$  with radius  $r > 0$ , and  $c$  is a positive constant such that whose value may change with each appearance.  $\mathcal{M}(\mathbb{R}^n)$  is the family of all complex or extended real-valued measurable functions on  $\mathbb{R}^n$ , and  $\mathcal{M}_0(\mathbb{R}^n)$  denote the family consisting of all those functions from  $\mathcal{M}(\mathbb{R}^n)$  which are finite. The symbol  $\mathcal{S}(\mathbb{R}^n)$  is the usual Schwartz space of infinitely differentiable rapidly decresing complex-valued functions on  $\mathbb{R}^n$ .

By  $\hat{\varphi}$  we denote the Fourier transform of  $\varphi \in \mathcal{S}(\mathbb{R}^n)$  in the version

$$\hat{\varphi}(x) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\xi} \varphi(\xi) d\xi, \quad x \in \mathbb{R}^n.$$

For two complex or extended real-valued measurable functions  $f, g$  on  $\mathbb{R}^n$ , the convolution  $f * g$  is given by

$$(f * g)(x) := \int_{\mathbb{R}^n} f(x - y) g(y) dy, \quad \text{for } x \in \mathbb{R}^n.$$

$\chi_B$  is the characteristic function of  $B$ , and  $\text{supp } f$  is the support of the function  $f$ , i.e. the closure of its zero set. The notation  $X \hookrightarrow Y$  denotes continuous embeddings from  $X$  to  $Y$ .

### 2.2 Modular spaces

The spaces studied in this paper are classified as semimodular spaces.

**Definition 1.** (Almeida & Hästö (2010)) Let  $X$  a vector space over  $\mathbb{R}$  or  $\mathbb{C}$ . A function  $\varrho : X \rightarrow [0, \infty]$  is called a semimodular on  $X$  if the following properties hold:

- $\varrho(0) = 0$ .
- $\varrho(\lambda f) = \varrho(f)$  for all  $f \in X$  and  $|\lambda| = 1$ .
- $\varrho(\lambda f) = 0$  for all  $\lambda > 0$  implies  $f = 0$ .
- $\lambda \mapsto \varrho(\lambda f)$  is left-continuous on  $[0, \infty)$  for every  $f \in X$ .

A semimodular  $\varrho$  is called a modular if  $\varrho(f) = 0$  implies  $f = 0$ .

A semimodular  $\varrho$  is called continuous if for every  $f \in X$  the mapping  $\lambda \mapsto \varrho(\lambda f)$  is continuous on  $[0, \infty)$ .

A semimodular  $\varrho$  is called quasiconvex if

$$\varrho(\theta f + (1 - \theta)g) \leq k[\theta\varrho(f) + (1 - \theta)\varrho(g)] \text{ for all } \theta \in [0, 1] \text{ and } f, g \in X.$$

Furthermore,  $k = 1$  in convex case and  $k \in [1, \infty)$  in the quasiconvex case. Where we have a semimodular, we obtain a normed space .

**Definition 2.** (Almeida & Hästö (2010)) If  $\varrho$  is a semimodular on  $X$ , then

$$X_\varrho := \{x \in X : \exists \lambda > 0, \varrho(\lambda x) < \infty\}$$

is called a semimodular space.

To show our main results, we shall employ the following theorem.

**Theorem 1.** (Almeida & Hästö (2010)) Let  $\varrho$  be a (quasi)convex semimodular on  $X$ . Then  $X_\varrho$  is a (quasi)normed space with the Luxemburg (quasi)norm given by

$$\|x\|_\varrho := \inf \left\{ \lambda > 0 : \varrho \left( \frac{1}{\lambda} x \right) \leq 1 \right\}.$$

From the definition and left-continuity of  $\varrho$  we get

$$\varrho(f) \leq 1 \quad \text{if and only if} \quad \|f\|_\varrho \leq 1.$$

### 2.3 Variable exponent Lebesgue spaces

We denote by  $\mathcal{P}(\mathbb{R}^n)$  the collection of all measurable functions  $p : \mathbb{R}^n \rightarrow (0, \infty]$  with  $p^- := \text{ess inf}_{x \in \mathbb{R}^n} p(x)$  and  $p^+ := \text{ess sup}_{x \in \mathbb{R}^n} p(x)$ . For exponents with  $p(x) \geq 1$  and complex or extended real-valued measurable functions  $f$  on  $\mathbb{R}^n$  a semimodular is defined by

$$\varrho_{p(\cdot)}(f) := \int_{\mathbb{R}^n} \phi_{p(x)}(|f(x)|) dx,$$

where

$$\phi_{p(x)}(t) := \begin{cases} t^{p(x)} & \text{if } p(x) \in (0, \infty), \\ 0 & \text{if } p(x) = \infty \text{ and } t \in [0, 1], \\ \infty & \text{if } p(x) = \infty \text{ and } t \in (1, \infty], \end{cases}$$

and the variable exponent Lebesgue space  $L_{p(\cdot)}(\mathbb{R}^n)$  is given by

$$L_{p(\cdot)}(\mathbb{R}^n) := \left\{ f : \mathbb{R}^n \rightarrow \mathbb{R} \text{ is measurable, } \int_{\mathbb{R}^n} |f(x)|^{p(x)} dx < \infty \right\}.$$

$L_{p(\cdot)}(\mathbb{R}^n)$  is a Banach space equipped with the norm

$$\|f\|_{L_{p(\cdot)}(\mathbb{R}^n)} := \inf \{\lambda > 0 : \varrho_{p(\cdot)}(f/\lambda) \leq 1\}.$$

Since the  $L_{p(\cdot)}$  does not have the same properties as  $L_p$ . Then, we assume the following standard conditions to ensure that the Hardy-Littlewood maximal operator  $M$  is bounded on  $L_{p(\cdot)}(\mathbb{R}^n)$ .

**Definition 3.** (Almeida & Hästö (2010)) Let  $h : \mathbb{R}^n \rightarrow \mathbb{R}$ .

i) We say that  $h$  is locally log-Hölder continuous,  $h \in C_{loc}^{\log}(\mathbb{R}^n)$ , if there exists a constant  $c_{\log} > 0$  with

$$|h(x) - h(y)| \leq \frac{c_{\log}}{\log\left(e + \frac{1}{|x-y|}\right)} \quad \text{for all } x, y \in \mathbb{R}^n.$$

ii) We say that  $h$  is globally log-Hölder continuous,  $h \in C^{\log}(\mathbb{R}^n)$ , if  $h \in C_{loc}^{\log}(\mathbb{R}^n)$  and there exist a  $h_\infty \in \mathbb{R}$  and a constant  $c_\infty > 0$  with

$$|h(x) - h_\infty| \leq \frac{c_\infty}{\log(e + |x|)} \quad \text{for all } x \in \mathbb{R}^n.$$

iii) We write  $h \in \mathcal{P}^{\log}(\mathbb{R}^n)$  if  $0 < h^- \leq h(x) \leq h^+ \leq \infty$  with  $1/h \in C^{\log}(\mathbb{R}^n)$ .

**Lemma 1.** (Almeida & Caetano (2020)) Let  $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ , then for all  $x \in \mathbb{R}^n$  and all  $r > 0$ , we have

$$\|\chi_{B(x,r)}\|_{L_{p(\cdot)}} \approx \begin{cases} r^{\frac{n}{p(x)}}, & \text{if } r \leq 1, \\ r^{\frac{n}{p_\infty}}, & \text{if } r \geq 1. \end{cases}$$

Moreover, we denote  $\frac{1}{p_\infty} := \left(\frac{1}{p}\right)_\infty$ .

**Lemma 2.** (Almeida & Caetano (2020)) Consider  $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$  with  $p(x) \geq 1$ , and  $\psi \in L_1$  and  $\psi_\varepsilon(x) := \varepsilon^{-n}\psi(x/\varepsilon)$ , for  $\varepsilon > 0$ . Suppose that  $\Psi(x) = \sup_{|y| \geq |x|} |\psi(y)|$  is integrable and  $f \in L_{p(\cdot)}$ . Then

$$\|\psi_\varepsilon * f\|_{L_{p(\cdot)}} \leq C \|\Psi\|_{L_1} \|f\|_{L_{p(\cdot)}},$$

where the constant  $C$  depends only on  $n$  and  $p$ .

## 2.4 Morrey spaces with variable exponents

Now, we give the definition of Morrey spaces  $M_{p(\cdot),u(\cdot)}(\mathbb{R}^n)$  which are a complement of  $L_{p(\cdot)}$ -spaces.

**Definition 4.** (Almeida & Caetano (2020)) Let  $p, u \in \mathcal{P}(\mathbb{R}^n)$  with  $0 < p^- \leq p(x) \leq u(x) \leq \infty$ , the variable exponent Morrey space  $M_{p(\cdot),u(\cdot)} := M_{p(\cdot),u(\cdot)}(\mathbb{R}^n)$  is the set of all  $f \in \mathcal{M}(\mathbb{R}^n)$  such that

$$\|f\|_{M_{p(\cdot),u(\cdot)}} = \sup_{x \in \mathbb{R}^n, r > 0} r^{\frac{n}{u(x)} - \frac{n}{p(x)}} \|f \chi_{B(x,r)}\|_{L_{p(\cdot)}} < \infty.$$

By the definition of the  $L_{p(\cdot)}$  quasinorm, we obtain

$$\|f\|_{M_{p(\cdot),u(\cdot)}} = \sup_{x \in \mathbb{R}^n, r > 0} \inf \left\{ \lambda > 0 : \varrho_{p(\cdot)} \left( r^{\frac{n}{u(x)} - \frac{n}{p(x)}} \frac{f}{\lambda} \chi_{B(x,r)} \right) \leq 1 \right\}.$$

Like in the  $L_{p(\cdot)}$  case, simple calculations show that (see Almeida & Caetano (2020))

$$\| |f|^t \|_{M_{p(\cdot)/t}, u(\cdot)/t} = \| f \|_{M_{p(\cdot), u(\cdot)}}^t, \quad t \in (0, \infty).$$

**Lemma 3.** (Almeida & Caetano (2020)) Let  $p \in \mathcal{P}(\mathbb{R}^n)$ ,  $\nu : \mathbb{R}^n \rightarrow [0, \infty)$  be measurable and  $g$  be a complex or extended real-valued function on  $\mathbb{R}^n \times \mathbb{R}^+ \times \mathbb{R}^n$  such that, for any  $(x, r) \in \mathbb{R}^n \times \mathbb{R}^+$ ,

$g(x, r, \cdot)$  is measurable on  $\mathbb{R}^n$ . Then

$$\begin{aligned} & \sup_{x \in \mathbb{R}^n, r > 0} \inf \left\{ \lambda > 0 : \varrho_{p(\cdot)} \left( \frac{g(x, r, \cdot)}{\lambda^{\nu(\cdot)}} \right) \leq 1 \right\} \\ &= \inf \left\{ \lambda > 0 : \sup_{x \in \mathbb{R}^n, r > 0} \varrho_{p(\cdot)} \left( \frac{g(x, r, \cdot)}{\lambda^{\nu(\cdot)}} \right) \leq 1 \right\}. \end{aligned}$$

Using Lemma 3 with  $\nu(y) \equiv 1$  and  $g(x, r, y) = r^{\frac{n}{u(x)} - \frac{n}{p(x)}} f(y) \chi_{B(x, r)}(y)$ , we get :

**Corollary 1.** Let  $p, u \in \mathcal{P}(\mathbb{R}^n)$  with  $p(x) \leq u(x)$ . For any  $f \in \mathcal{M}(\mathbb{R}^n)$  it holds

$$\| f \|_{M_{p(\cdot), u(\cdot)}} = \inf \left\{ \lambda > 0 : \sup_{x \in \mathbb{R}^n, r > 0} \varrho_{p(\cdot)} \left( \frac{1}{\lambda} r^{\frac{n}{u(x)} - \frac{n}{p(x)}} f \chi_{B(x, r)} \right) \leq 1 \right\}.$$

**Remark 1.** It was observed in Almeida & Caetano (2020) that

$$\varrho_{p(\cdot), u(\cdot)}(f) := \sup_{x \in \mathbb{R}^n, r > 0} \varrho_{p(\cdot)} \left( r^{\frac{n}{u(x)} - \frac{n}{p(x)}} f \chi_{B(x, r)} \right).$$

**Lemma 4.** (Almeida & Caetano (2020)) Let  $p \in \mathcal{P}(\mathbb{R}^n)$ , then for all  $g \in \mathcal{M}(\mathbb{R}^n)$ ,

$$\sup_{x \in \mathbb{R}^n, r > 0} \varrho_{p(\cdot)}(g \chi_{B(x, r)}) = \varrho_{p(\cdot)}(g).$$

**Remark 2.** Let  $p, u \in \mathcal{P}(\mathbb{R}^n)$ , if  $p(\cdot) = u(\cdot)$  we have  $M_{p(\cdot), p(\cdot)} = L_{p(\cdot)}$  (with equal quasinorms).

## 2.5 Mixed Lebesgue-sequence spaces

First, we present the semimodular of mixed Lebesgue-sequence spaces.

**Definition 5.** Let  $p, q \in \mathcal{P}(\mathbb{R}^n)$ , the semimodular of mixed Lebesgue-sequence space  $\ell_{q(\cdot)}(L_{p(\cdot)})$  is defined by

$$\varrho_{\ell_{q(\cdot)}(L_{p(\cdot)})}((f_v)_v) := \sum_{v \geq 0} \inf \left\{ \lambda_v > 0 \mid \varrho_{p(\cdot)} \left( f_v / \lambda_v^{\frac{1}{q(\cdot)}} \right) \leq 1 \right\}.$$

With the convention  $\lambda^{1/\infty} = 1$ . Also the norm is defined as usual:

$$\|(f_v)_v\|_{\ell_{q(\cdot)}(L_{p(\cdot)})} := \inf \left\{ \mu > 0 \mid \varrho_{\ell_{q(\cdot)}(L_{p(\cdot)})} \left( \frac{1}{\mu} (f_v)_v \right) \leq 1 \right\}.$$

If  $q^+ < \infty$ , then

$$\inf \left\{ \lambda > 0 \mid \varrho_{p(\cdot)} \left( f / \lambda^{\frac{1}{q(\cdot)}} \right) \leq 1 \right\} = \left\| |f|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}}.$$

Which gives

$$\varrho_{\ell_{q(\cdot)}(L_{p(\cdot)})}((f_v)_v) = \sum_{v \geq 0} \left\| |f_v|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}}.$$

The left-continuity of the semimodular implies that

$$\|(f_v)_v\|_{\ell_{q(\cdot)}(L_{p(\cdot)})} \leq 1 \text{ if and only if } \varrho_{\ell_{q(\cdot)}(L_{p(\cdot)})}((f_v)_v) \leq 1 \text{ (unit ball property).}$$

$\ell_{q(\cdot)}(L_{p(\cdot)})$  is a really iterated space when  $q \in (0, \infty]$  (See Almeida & Hästö (2010)), and in that case the quasinorm is given by

$$\|(f_v)_v\|_{\ell_q(L_{p(\cdot)})} = \left\| \left( \|f_v\|_{L_{p(\cdot)}} \right)_v \right\|_{\ell_q}.$$

## 2.6 Variable exponent mixed Morrey-sequence spaces

We recall the definition of semimodular of mixed Morrey-sequence spaces.

**Definition 6.** (Almeida & Caetano (2020)) Let  $p, q, u \in \mathcal{P}(\mathbb{R}^n)$  with  $p(x) \leq u(x)$ , and  $(f_v)_v \subset \mathcal{M}(\mathbb{R}^n)$ , then

$$\varrho_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})}((f_v)_v) := \sum_{v \geq 0} \sup_{x \in \mathbb{R}^n, r > 0} \inf \left\{ \lambda > 0 : \varrho_{p(\cdot)} \left( r^{\frac{n}{u(x)} - \frac{n}{p(x)}} f_v \chi_{B(x,r)} / \lambda^{\frac{1}{q(\cdot)}} \right) \leq 1 \right\}. \quad (1)$$

**Remark 3.** When  $q^+ < \infty$  or  $q^+ = \infty$  and  $p(x) \geq q(x)$ , we can simplify (1) like in the  $\ell_{q(\cdot)}(L_{p(\cdot)})$  case:

$$\varrho_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})}((f_v)_v) = \sum_{v \geq 0} \sup_{x \in \mathbb{R}^n, r > 0} \left\| \phi_{q(\cdot)} \left( r^{\frac{n}{u(x)} - \frac{n}{p(x)}} |f_v| \chi_{B(x,r)} \right) \right\|_{L_{\frac{p(\cdot)}{q(\cdot)}}}.$$

Below, we give the mixed Morrey-sequence spaces.

**Definition 7.** (Almeida & Caetano (2020)) Let  $p, q, u \in \mathcal{P}(\mathbb{R}^n)$  with  $p(x) \leq u(x)$ . The mixed Morrey-sequence space  $\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})$  is the set of all sequences  $(f_v)_v \subset \mathcal{M}(\mathbb{R}^n)$  such that  $\varrho_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})}(\mu(f_v)_v) < \infty$  for some  $\mu > 0$ .

For  $(f_v)_v \subset \ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})$  we define

$$\|(f_v)_v\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})} := \inf \left\{ \mu > 0 : \varrho_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})} \left( \frac{1}{\mu} (f_v)_v \right) \leq 1 \right\}.$$

As in the case of the variable Lebesgue-sequence spaces we have the following proposition when  $q$  is a constant.

**Proposition 1.** (Almeida & Caetano (2020)) Let  $p, q, u \in \mathcal{P}(\mathbb{R}^n)$  with  $p(x) \leq u(x)$ . If  $q \in (0, \infty]$  is a constant (almost everywhere), then

$$\|(f_v)_v\|_{\ell_q(M_{p(\cdot),u(\cdot)})} = \left\| \left( \|f_v\|_{M_{p(\cdot),u(\cdot)}} \right)_v \right\|_{\ell_q},$$

for every sequence  $(f_v)_v \subset \mathcal{M}(\mathbb{R}^n)$ .

## 2.7 2-microlocal Besov-Morrey spaces and 2-microlocal Triebel-Lizorkin-Morrey spaces

To define 2-microlocal Besov-Morrey spaces and 2-microlocal Triebel-Lizorkin-Morrey spaces, we need some general definitions, already seen in the constant exponents case.

**Definition 8.** (Almeida & Hästö (2010)) We say a pair  $(\varphi, \Phi) \in \mathcal{S}$  satisfy

- $\text{supp } \hat{\varphi} \subseteq \{\xi \in \mathbb{R}^n : \frac{1}{2} \leq |\xi| \leq 2\}$  and  $|\hat{\varphi}(\xi)| \geq c > 0$  when  $\frac{3}{5} \leq |\xi| \leq \frac{5}{3}$ .
- $\text{supp } \hat{\Phi} \subseteq \{\xi \in \mathbb{R}^n : |\xi| \leq 2\}$  and  $|\hat{\Phi}(\xi)| \geq c > 0$  when  $|\xi| \leq \frac{5}{3}$ .

We set  $\varphi_v(x) := 2^{vn}\varphi(2^v x)$  for  $v \in \mathbb{N}$  and  $\varphi_0(x) := \Phi(x)$ .

**Definition 9.** (Almeida & Caetano (2020)) Let  $\beta_1 \leq \beta_2$  and  $\beta \geq 0$  be real numbers. The class of admissible weight sequences  $\mathcal{W}_{\beta_1, \beta_2}^\beta(\mathbb{R}^n)$  is the collection of all sequences  $\mathbf{w} = (w_v)_{v \in \mathbb{N}_0}$  of measurable functions  $w_v$  on  $\mathbb{R}^n$  such that

- i) There exists a constant  $C > 0$  such that

$$0 < w_v(x) \leq Cw_v(x) \leq Cw_v(y)(1 + 2^v|x - y|)^\beta \text{ for } x, y \in \mathbb{R}^n \text{ and } v \in \mathbb{N}_0;$$

- ii) For all  $x \in \mathbb{R}^n$  and  $v \in \mathbb{N}_0$

$$2^{\beta_1}w_v(x) \leq w_{v+1}(x) \leq 2^{\beta_2}w_v(x).$$

The spaces  $B_{p(\cdot), q(\cdot)}^{\mathbf{w}}(\mathbb{R}^n)$  with variable integrability have been studied in the general setting of 2-microlocal spaces, see Almeida & Caetano (2016); Kempka (2009, 2010). The 2-microlocl Besov-Morrey spaces  $\mathcal{N}_{p(\cdot), u(\cdot), q(\cdot)}^{\mathbf{w}}(\mathbb{R}^n)$  and 2-microlocal Triebel-Lizorkin-Morrey spaces  $\mathcal{E}_{p(\cdot), q(\cdot)}^{\mathbf{w}, u(\cdot)}(\mathbb{R}^n)$  have recently been studied by Besov (2003) and Almeida & Caetano (2020) respectively. Usually, These spaces are defined by using the functions  $\varphi_v$ .

**Definition 10.** (Almeida & Caetano (2020)) Consider an admissible system  $\{\varphi_v\}$ . Let  $\mathbf{w} \in \mathcal{W}_{\beta_1, \beta_2}^\beta(\mathbb{R}^n)$  be admissible weights and let  $p, q \in \mathcal{P}^{\log}(\mathbb{R}^n)$  and  $u \in \mathcal{P}(\mathbb{R}^n)$  with  $0 < p^- \leq p(x) \leq u(x) \leq \infty$ .

We define  $\mathcal{N}_{p(\cdot), u(\cdot), q(\cdot)}^{\mathbf{w}}$  as the collection of all  $f \in \mathcal{S}'(\mathbb{R}^n)$  such that

$$\|f\|_{\mathcal{N}_{p(\cdot), u(\cdot), q(\cdot)}^{\mathbf{w}}} := \|w_v (\varphi_v * f)_v\|_{\ell_{q(\cdot)}(M_{p(\cdot), u(\cdot)})} < \infty.$$

In particular case  $w_v(x) = 2^{v\alpha(x)}$ , with  $\alpha \in C_{loc}^{\log}(\mathbb{R}^n)$ , we have Besov-Morrey spaces with variable exponents, and we write in that case

$$\mathcal{N}_{p(\cdot), u(\cdot), q(\cdot)}^{\mathbf{w}} = \mathcal{N}_{p(\cdot), u(\cdot), q(\cdot)}^{\alpha(\cdot)}.$$

The Besov-Morrey spaces can be associated with the following semi-modular:

$$\varrho_{\mathcal{N}_{p(\cdot), u(\cdot), q(\cdot)}^{\alpha(\cdot)}}(f) := \varrho_{\ell_{q(\cdot)}(M_{p(\cdot), u(\cdot)})}((2^{v\alpha(\cdot)}\varphi_v * f)_v). \quad (2)$$

**Remark 4.** It was shown in Almeida & Caetano (2020) that

$$\mathcal{S} \hookrightarrow \mathcal{N}_{p(\cdot), u(\cdot), q(\cdot)}^{\mathbf{w}} \hookrightarrow \mathcal{S}'.$$

**Definition 11.** Let  $(\varphi_v)_{v \in \mathbb{N}_0}$  be constructed as in Definition 8 and  $\mathbf{w} \in \mathcal{W}_{\beta_1, \beta_2}^\beta(\mathbb{R}^n)$  be admissible weights. Let  $p, q \in \mathcal{P}^{\log}(\mathbb{R}^n)$  and  $u \in \mathcal{P}(\mathbb{R}^n)$  with  $0 < p^- \leq p(x) \leq u(x) \leq \sup u < \infty$  and  $q^-, q^+ \in (0, \infty)$ . Then

$$\mathcal{E}_{p(\cdot), q(\cdot)}^{\mathbf{w}, u(\cdot)}(\mathbb{R}^n) := \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{\mathcal{E}_{p(\cdot), q(\cdot)}^{\mathbf{w}, u(\cdot)}(\mathbb{R}^n)} < \infty \right\}$$

where

$$\|f\|_{\mathcal{E}_{p(\cdot), q(\cdot)}^{\mathbf{w}, u(\cdot)}} := \|w_v (\varphi_v * f)_v\|_{M_{p(\cdot), u(\cdot)}(\ell_{q(\cdot)})} := \left\| \left( \sum_{v=0}^{\infty} |w_v \varphi_v * f|^{q(\cdot)} \right)^{1/q(\cdot)} \right\|_{M_{p(\cdot), u(\cdot)}}.$$

In particular case  $w_v(x) = 2^{v\alpha(x)}$ , with  $\alpha \in C_{loc}^{\log}(\mathbb{R}^n)$ , we have Triebel-Lizorkin-Morrey spaces with variable exponents, and we write in that case

$$\mathcal{E}_{p(\cdot),q(\cdot)}^{w,u(\cdot)}(\mathbb{R}^n) = \mathcal{E}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),u(\cdot)}(\mathbb{R}^n).$$

We can also associate the following semimodular to the Triebel-Lizorkin-Morrey spaces.

$$\varrho_{\mathcal{E}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),u(\cdot)}}(f) := \varrho_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})}((2^{v\alpha(\cdot)}\varphi_v * f)_v).$$

**Remark 5.** It was shown in Besov (2003) that  $\mathcal{E}_{p(\cdot),q(\cdot)}^{w,u(\cdot)}$  are quasi-normed spaces, and

$$\mathcal{S} \hookrightarrow \mathcal{E}_{p(\cdot),q(\cdot)}^{w,u(\cdot)} \hookrightarrow \mathcal{S}'.$$

### 3 Main results

In this section, we shall present our first main result that establishes the basic embeddings between Besov-Morrey spaces and Triebel-Lizorkin-Morrey spaces.

**Theorem 2.** Let  $\alpha, \alpha_0, \alpha_1 \in L^\infty \cap C_{loc}^{\log}(\mathbb{R}^n)$  and  $u, p, q_0, q_1 \in \mathcal{P}(\mathbb{R}^n)$ , with  $0 < p^- \leq p(x) \leq u(x) \leq \infty$ .

(i) If  $q_0 \leq q_1$ , then

$$\mathcal{N}_{p(\cdot),u(\cdot),q_0(\cdot)}^{\alpha(\cdot)} \hookrightarrow \mathcal{N}_{p(\cdot),u(\cdot),q_1(\cdot)}^{\alpha(\cdot)}.$$

(ii) If  $(\alpha_0 - \alpha_1)^- > 0$ , then

$$\mathcal{N}_{p(\cdot),u(\cdot),q_0(\cdot)}^{\alpha_0(\cdot)} \hookrightarrow \mathcal{N}_{p(\cdot),u(\cdot),q_1(\cdot)}^{\alpha_1(\cdot)}.$$

(iii) If  $p^+, q^+ < \infty$ , then

$$\mathcal{N}_{p(\cdot),u(\cdot),\min\{p(\cdot),q(\cdot)\}}^{\alpha(\cdot)} \hookrightarrow \mathcal{E}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),u(\cdot)} \hookrightarrow \mathcal{N}_{p(\cdot),u(\cdot),\max\{p(\cdot),q(\cdot)\}}^{\alpha(\cdot)}.$$

*Proof.* For (i) it suffices to show that  $\varrho_{\mathcal{N}_{p(\cdot),u(\cdot),q_0(\cdot)}^{\alpha(\cdot)}}(f/\mu) \geq \varrho_{\mathcal{N}_{p(\cdot),u(\cdot),q_1(\cdot)}^{\alpha(\cdot)}}(f/\mu)$  for every  $\mu > 0$ .

Let  $\mu > 0$ , we have  $q_0 \leq q_1$  then  $\lambda^{\frac{1}{q_0(\cdot)}} \leq \lambda^{\frac{1}{q_1(\cdot)}}$ , when  $\lambda \leq 1$ . By using the equality (2), Lemma 3 and

Lemma 4, one reaches

$$\varrho_{\mathcal{N}_{p(\cdot),u(\cdot),q_0(\cdot)}^{\alpha(\cdot)}}(f/\mu) \geq \varrho_{\mathcal{N}_{p(\cdot),u(\cdot),q_1(\cdot)}^{\alpha(\cdot)}}(f/\mu),$$

which gives (i).

(ii) From (i), one obtains,  $\mathcal{N}_{p(\cdot),u(\cdot),q_0(\cdot)}^{\alpha_0(\cdot)} \hookrightarrow \mathcal{N}_{p(\cdot),u(\cdot),q_0^+(\cdot)}^{\alpha_0(\cdot)}$  and  $\mathcal{N}_{p(\cdot),u(\cdot),q_1^-(\cdot)}^{\alpha_1(\cdot)} \hookrightarrow \mathcal{N}_{p(\cdot),u(\cdot),q_1(\cdot)}^{\alpha_1(\cdot)}$ .

Then, it suffices to prove that  $\mathcal{N}_{p(\cdot),u(\cdot),q_0^+(\cdot)}^{\alpha_0(\cdot)} \hookrightarrow \mathcal{N}_{p(\cdot),u(\cdot),q_1^-(\cdot)}^{\alpha_1(\cdot)}$ .

Observe that,

$$\begin{aligned} \|f\|_{\mathcal{N}_{p(\cdot),u(\cdot),q_1^-(\cdot)}^{\alpha_1(\cdot)}} &= \left\| 2^{v\alpha_1(\cdot)} \varphi_v * f \right\|_{l^{q_1^-}(M_{p(\cdot),u(\cdot)})} \\ &= \left\| \left\| 2^{v\alpha_1(\cdot)} \varphi_v * f \right\|_{M_{p(\cdot),u(\cdot)}} \right\|_{\ell^{q_1^-}} \\ &\leq c \left\| \left\| 2^{v\alpha_0(\cdot)} \varphi_v * f \right\|_{M_{p(\cdot),u(\cdot)}} \right\|_{\ell^\infty} \\ &\leq c \left\| \left\| 2^{v\alpha_0(\cdot)} \varphi_v * f \right\|_{M_{p(\cdot),u(\cdot)}} \right\|_{\ell^{q_0^+}} \\ &\leq c \|f\|_{\mathcal{N}_{p(\cdot),u(\cdot),q_0^+(\cdot)}^{\alpha_0(\cdot)}}, \end{aligned}$$

with  $c = (\sum_{v \geq 0} 2^{-vq_1^-}(\alpha_0 - \alpha_1)^-)^\frac{1}{q_1^-} < \infty$ .

(iii) Let us first show the embedding  $\mathcal{N}_{p(\cdot), u(\cdot), \min\{p(\cdot), q(\cdot)\}}^{\alpha(\cdot)} \hookrightarrow \mathcal{E}_{p(\cdot), q(\cdot)}^{\alpha(\cdot), u(\cdot)}$ .

Put  $t(\cdot) := \min\{p(\cdot), q(\cdot)\}$  and  $f_v(x) = 2^{v\alpha(x)} |\varphi_v * f(x)|$ .

We assume that  $\varrho_{\mathcal{N}_{p(\cdot), u(\cdot), t(\cdot)}^{\alpha(\cdot)}}(f) \leq 1$  and let us prove that  $\varrho_{\mathcal{E}_{p(\cdot), q(\cdot)}^{\alpha(\cdot), u(\cdot)}}(f) \leq c$ .

Since  $\ell^{t(x)} \hookrightarrow \ell^{q(x)}$ , we get

$$\begin{aligned} \varrho_{\mathcal{E}_{p(\cdot), q(\cdot)}^{\alpha(\cdot), u(\cdot)}}(f) &= \varrho_{M_{p(\cdot), u(\cdot)}}(||f_v||_{\ell^{q(\cdot)}}) \\ &\leq \varrho_{M_{p(\cdot), u(\cdot)}}(||f_v||_{\ell^{t(\cdot)}}) \\ &\leq \sup_{x \in \mathbb{R}^n, r > 0} \varrho_{\frac{p(\cdot)}{t(\cdot)}} \left( r^{(\frac{n}{u(x)} - \frac{n}{p(x)})t(\cdot)} \sum_{v \geq 0} f_v^{t(\cdot)} \chi_{B(x, r)} \right). \end{aligned} \quad (3)$$

We know that  $\varrho_{\frac{p(\cdot)}{t(\cdot)}}(f) \leq 1$  if and only if  $||f||_{\frac{p(\cdot)}{t(\cdot)}} \leq 1$ , then it suffices to show that the inequality (3) is bounded by a constant.

We have

$$\begin{aligned} \sup_{x \in \mathbb{R}^n, r > 0} \left\| r^{(\frac{n}{u(x)} - \frac{n}{p(x)})t(\cdot)} \sum_{v \geq 0} f_v^{t(\cdot)} \chi_{B(x, r)} \right\|_{\frac{p(\cdot)}{t(\cdot)}} &\leq \sum_{v \geq 0} \sup_{x \in \mathbb{R}^n, r > 0} \left\| \phi_{t(\cdot)} \left( r^{\frac{n}{u(x)} - \frac{n}{p(x)}} f_v \chi_{B(x, r)} \right) \right\|_{\frac{p(\cdot)}{t(\cdot)}} \\ &= \varrho_{\mathcal{N}_{p(\cdot), u(\cdot), t(\cdot)}^{\alpha(\cdot)}}(f) \\ &\leq 1, \end{aligned}$$

which gives the result.

For the second embedding in (iii)

$$\mathcal{E}_{p(\cdot), q(\cdot)}^{\alpha(\cdot), u(\cdot)} \hookrightarrow \mathcal{N}_{p(\cdot), u(\cdot), \max\{p(\cdot), q(\cdot)\}}^{\alpha(\cdot)},$$

let  $s(\cdot) = \max\{p(\cdot), q(\cdot)\}$ . We assume that  $\varrho_{\mathcal{E}_{p(\cdot), q(\cdot)}^{\alpha(\cdot), u(\cdot)}}(f) \leq 1$  and we prove that  $\varrho_{\mathcal{N}_{p(\cdot), s(\cdot)}^{\alpha(\cdot), u(\cdot)}}(f) \leq c$ .

Using the reverse triangle inequality ( $p/s \leq 1$ ), we obtain

$$\begin{aligned} \varrho_{\mathcal{N}_{p(\cdot), u(\cdot), s(\cdot)}^{\alpha(\cdot)}}(f) &= \varrho_{\ell^{s(\cdot)}(M_{p(\cdot), u(\cdot)})}(f) \\ &= \sum_{v \geq 0} \sup_{x \in \mathbb{R}^n, r > 0} \left\| \phi_{s(\cdot)} \left( r^{\frac{n}{u(x)} - \frac{n}{p(x)}} f_v \chi_{B(x, r)} \right) \right\|_{\frac{p(\cdot)}{s(\cdot)}} \\ &\leq \sup_{x \in \mathbb{R}^n, r > 0} \left\| r^{(\frac{n}{u(x)} - \frac{n}{p(x)})s(\cdot)} \sum_{v \geq 0} |f_v|^{s(\cdot)} \chi_{B(x, r)} \right\|_{\frac{p(\cdot)}{s(\cdot)}}. \end{aligned}$$

Then it suffices to show that  $\sup_{x \in \mathbb{R}^n, r > 0} \varrho_{\frac{p(\cdot)}{s(\cdot)}} \left( r^{(\frac{n}{u(x)} - \frac{n}{p(x)})s(\cdot)} \sum_{v \geq 0} |f_v|^{s(\cdot)} \chi_{B(x, r)} \right)$  is bounded.

$$\begin{aligned} \sup_{x \in \mathbb{R}^n, r > 0} \varrho_{\frac{p(\cdot)}{s(\cdot)}} \left( r^{(\frac{n}{u(x)} - \frac{n}{p(x)})s(\cdot)} \sum_{v \geq 0} |f_v|^{s(\cdot)} \chi_{B(x, r)} \right) &= \sup_{x \in \mathbb{R}^n, r > 0} \varrho_{p(\cdot)} \left( r^{(\frac{n}{u(x)} - \frac{n}{p(x)})} \left( \sum_{v \geq 0} ||f_v||^{s(\cdot)} \right)^{1/s(\cdot)} \chi_{B(x, r)} \right) \\ &= \varrho_{M_{p(\cdot), u(\cdot)}}(||f_v||_{\ell^{s(\cdot)}}) \\ &\leq \varrho_{M_{p(\cdot), u(\cdot)}}(||f_v||_{\ell^{q(\cdot)}}) \\ &= \varrho_{\mathcal{E}_{p(\cdot), q(\cdot)}^{\alpha(\cdot), u(\cdot)}}(f) \\ &\leq 1. \end{aligned}$$

□

Now, we are in the position to present the Sobolev-type embeddings in the Besov-Morrey spaces when the radius of the ball  $B(x, r)$  satisfies  $0 < r \leq 1$ .

The following lemma is very useful in proving the next theorem.

**Lemma 5.** (*Almeida & Hästö (2010)*) Let  $p_1, p_0, q \in \mathcal{P}(\mathbb{R}^n)$  with  $\alpha = n/p_1$  and  $1/q$  locally log-Hölder continuous. If  $p_1 \geq p_0$ , then there exists  $c > 0$  such that

$$\left\| \left| c2^{v\alpha(\cdot)} g \right|^{q(\cdot)} \right\|_{\frac{p_1(\cdot)}{q(\cdot)}} \leq \left\| \left| 2^{v(\alpha(\cdot) + \frac{n}{p_0(\cdot)} - \frac{n}{p_1(\cdot)})} g \right|^{q(\cdot)} \right\|_{\frac{p_0(\cdot)}{q(\cdot)}} + 2^{-v},$$

for all  $v \in \mathbb{N}_0$  and  $g \in L^{p_0(\cdot)} \cap \mathcal{S}'$  with  $\text{supp } \hat{g} \subset \{\xi : |\xi| \leq 2^{v+1}\}$  such that the norm on the right-hand side is at most one.

The following theorem assures the the Sobolev-type embeddings.

**Theorem 3.** Let  $p_0, p_1, u_0, u_1, q \in \mathcal{P}(\mathbb{R}^n)$  such that  $0 < p_0^- \leq p_0(x) \leq u_0(x) \leq \infty, 0 < p_1^- \leq p_1(x) \leq u_1(x) \leq \infty, u_1(x) \leq u_0(x)$  and  $\alpha_0, \alpha_1 \in L^\infty \cap C_{loc}^{\log}(\mathbb{R}^n)$  with  $\alpha_0 \geq \alpha_1$ .

If  $\frac{1}{q}$  and  $\alpha_0(x) - \frac{n}{p_0(x)} = \alpha_1(x) - \frac{n}{p_1(x)}$  are locally log-Hölder continuous, then

$$\mathcal{N}_{p_0(\cdot), u_0(\cdot), q(\cdot)}^{\alpha_0(\cdot)} \hookrightarrow \mathcal{N}_{p_1(\cdot), u_1(\cdot), q(\cdot)}^{\alpha_1(\cdot)}.$$

*Proof.* Suppose without loss of generality that  $\varrho_{\mathcal{N}_{p_0(\cdot), u_0(\cdot), q(\cdot)}^{\alpha_0(\cdot)}}(f) \leq 1$ . So, we have to show that

$$\varrho_{\mathcal{N}_{p_1(\cdot), u_1(\cdot), q(\cdot)}^{\alpha_1(\cdot)}}(f) \leq c.$$

By using the equality (2), one obtains

$$\begin{aligned} \varrho_{\mathcal{N}_{p_1(\cdot), u_1(\cdot), q(\cdot)}^{\alpha_1(\cdot)}}(f) &= \varrho_{\ell^{q(\cdot)}(M_{p_1(\cdot), u_1(\cdot)})} \left( 2^{v\alpha_1(\cdot)} \varphi_v * f \right) \\ &= \sum_{v \geq 0} \sup_{x \in \mathbb{R}^n, r > 0} \left\| \phi_{q(\cdot)} \left( r^{\frac{n}{u_1(x)} - \frac{n}{p_1(x)}} 2^{v\alpha_1(\cdot)} \varphi_v * f \chi_{B(x,r)} \right) \right\|_{\frac{p_1(\cdot)}{q(\cdot)}} \\ &= \sum_{v \geq 0} \sup_{x \in \mathbb{R}^n, r > 0} \left\| \left| r^{\frac{n}{u_1(x)} - \frac{n}{p_1(x)}} 2^{v\alpha_1(\cdot)} \varphi_v * f \right|^{q(\cdot)} \right\|_{L_{\frac{p_1(\cdot)}{q(\cdot)}}(B(x,r))}. \end{aligned}$$

Applying Lemma 5 with  $\alpha(\cdot) = \alpha_1(\cdot)$  and  $g = r^{\frac{n}{u_1(x)} - \frac{n}{p_1(x)}} \varphi_v * f$ , we get

$$\begin{aligned} &\sum_{v \geq 0} \sup_{x \in \mathbb{R}^n, r > 0} \left\| \left| r^{\frac{n}{u_1(x)} - \frac{n}{p_1(x)}} 2^{v\alpha_1(\cdot)} \varphi_v * f \right|^{q(\cdot)} \right\|_{L_{\frac{p_1(\cdot)}{q(\cdot)}}(B(x,r))} \\ &\leq \sum_{v \geq 0} \sup_{x \in \mathbb{R}^n, r > 0} \left\| \phi_{q(\cdot)} \left( r^{\frac{n}{u_1(x)} - \frac{n}{p_1(x)}} 2^{v\alpha_0(\cdot)} \varphi_v * f \chi_{B(x,r)} \right) \right\|_{\frac{p_0(\cdot)}{q(\cdot)}} + \sum_{v \geq 0} 2^{-v}. \end{aligned}$$

Since  $0 < r \leq 1$  and  $u_1(x) \leq u_0(x)$ , one obtains

$$\begin{aligned} &\sum_{v \geq 0} \sup_{x \in \mathbb{R}^n, r > 0} \left\| \phi_{q(\cdot)} \left( r^{\frac{n}{u_1(x)} - \frac{n}{p_1(x)}} 2^{v\alpha_0(\cdot)} \varphi_v * f \chi_{B(x,r)} \right) \right\|_{\frac{p_0(\cdot)}{q(\cdot)}} + \sum_{v \geq 0} 2^{-v} \\ &\leq \sum_{v \geq 0} \sup_{x \in \mathbb{R}^n, r > 0} \left\| \phi_{q(\cdot)} \left( r^{\frac{n}{u_0(x)} - \frac{n}{p_0(x)}} 2^{v\alpha_0(\cdot)} \varphi_v * f \chi_{B(x,r)} \right) \right\|_{\frac{p_0(\cdot)}{q(\cdot)}} + \sum_{v \geq 0} 2^{-v} \\ &\leq \varrho_{\mathcal{N}_{p_0(\cdot), u_0(\cdot), q(\cdot)}^{\alpha_0(\cdot)}}(f) + \sum_{v \geq 0} 2^{-v} \\ &\leq 1 + \sum_{v \geq 0} 2^{-v} \\ &\leq c. \end{aligned}$$

The proof is complete.  $\square$

**Corollary 2.** Let  $p_0, p_1, u_0, u_1, q_0, q_1 \in \mathcal{P}(\mathbb{R}^n)$  such that  $0 < p_0^- \leq p_0(x) \leq u_0(x) \leq \infty, 0 < p_1^- \leq p_1(x) \leq u_1(x) \leq \infty$  and  $\alpha_0, \alpha_1 \in L^\infty \cap C_{loc}^{log}(\mathbb{R}^n)$  such that  $\alpha_0 \geq \alpha_1$ .

If

$$\alpha_0(x) - \frac{n}{p_0(x)} = \alpha_1(x) - \frac{n}{p_1(x)} + \varepsilon(x)$$

is locally log-Hölder continuous, and  $\varepsilon^- > 0$ , then

$$\mathcal{N}_{p_0(\cdot), u_0(\cdot), q_0(\cdot)}^{\alpha_0(\cdot)} \hookrightarrow \mathcal{N}_{p_1(\cdot), u_1(\cdot), q_1(\cdot)}^{\alpha_1(\cdot)}.$$

*Proof.* By using Theoreme 2 (i), we get

$$\mathcal{N}_{p_0(\cdot), u_0(\cdot), q_0(\cdot)}^{\alpha_0(\cdot)} \hookrightarrow \mathcal{N}_{p_0(\cdot), u_0(\cdot), \infty}^{\alpha_0(\cdot)},$$

and from Theoreme 3 we have

$$\mathcal{N}_{p_0(\cdot), u_0(\cdot), \infty}^{\alpha_0(\cdot)} \hookrightarrow \mathcal{N}_{p_1(\cdot), u_1(\cdot), \infty}^{\alpha_1(\cdot) + \varepsilon(\cdot)}.$$

An application of (ii) in Theoreme 3 implies that  $\mathcal{N}_{p_1(\cdot), u_1(\cdot), \infty}^{\alpha_1(\cdot) + \varepsilon(\cdot)} \hookrightarrow \mathcal{N}_{p_1(\cdot), u_1(\cdot), q(\cdot)}^{\alpha_1(\cdot)}$ , which finishes the proof.  $\square$

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